1 Bell's Inequality for Non-Physicists

In the book "Dance of the Photons: From Einstein to Quantum Teleportation", Anton Zeilinger gives what he calls "Bell's inequality for non-physicists", to "make Bells' inequality easily accessible for the general readership". Although it is given in a story-book form, I believe Zeilinger intends that it be mathematically rigorous - not just intended to give the "spirit" of the inequality.

For the book I am writing, which has an intended audience of Junior-High students and up, I was in the process of more fully explaining and re-writing Zeilinger's exposition (but not changing any of the logic given in his exposition). However, I came to a point where I think he made a fundamental error (not sure if it is best to say it is a mathematical, or a modeling error). It can be fixed, but it makes the story much clunkier, and may demand more sophistication on the part of the reader. So below, I present what I had written, and point out where I think the argument goes wrong. Comments are welcome.

2 A Story-Book Town

In a (strange) town, each person is described based on their height (either Tall or Short); on their eye color (either Blue or Green); and on their hair color (either Red or White). So, there are $2^3 = 8$ combinations of descriptions, meaning there are eight different types of people (based on these descriptions).

As an example, suppose the number of people of each description is given in the following table (where T stands for Tall, S for Short, B for Blue-eyed, G for Green-eyed, R for Red hair, and W for White hair):

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TBR  16
TBW   6
TGR  18
TGW  14
SBR  18
SBW   8
SGR  12
SGW   8
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You might notice that all of the numbers are even. There is a reason for that.

In this town, the people are divided up into *couples* (pairs), with a very *odd* property. In any couple, the descriptions of the two people in the couple are exactly the same (descriptions based on the three attributes of height, eye color, and hair color). For example, if one person in a couple is Tall, with Green eyes, and White hair, then so is the other person in the couple.[1] The fact that the two people in any couple have the same descriptions illustrates the maxim that "likes attract likes".

So, in the town, the number of couples of each description is given in the following table:

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TBR 8
TBW 3
TGR 9
TGW 7
SBR 9
SBW 4
SGR 6
SGW 4
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It is important to realize that each couple is included in *exactly* one of the eight subsets.

Since the two people in each couple have identical descriptions, we can refer to a "Tall couple" or a "Green-eyed couple", or a "White-haired couple", in the cases that both people are Tall, or both have Green eyes, or both have White hair, etc. Similarly, we can refer to a couple "with Red hair".

Some basic questions As a warm-up to what will follow, let's ask and answer some simple questions about the couples.

How many couples are there in this example?

[1] In the exposition in Zeilinger, the town consists of identical *twins*. I have changed "twins" to "couples" because in English there is an ambiguity in the use of the phrase "pairs of twins" that may cause confusion - it caused me confusion. For example, the word "twin" (as in "are you a twin?") refers to one person, and the word "twins" (as in "are you twins?"), normally refers to two people. So, what then is a "pair of twins"? Does a "pair of twins" refer to two people or to four people? Logically, it should be four, but in normal speech it is two. Hence, the ambiguity. But it is clear that "person", "couple" and "pair of couples" refers to one, two and four people, respectively. No ambiguity.

Answer: 50 (I swear I made up these numbers without any actual thought - so it is interesting that the sum came out to such a nice number).

How many Tall couples are there?

Answer: 27

How many Blue-eyed couples are there?

Answer: 24

How many Tall, Blue-eyed couples are there?

Answer: 11

Ok, so now that we have some understanding of how these subsets and numbers work, let's make an obvious statement (implicit in the answers already given):

> **The basic equality** The number of Tall, Blue-eyed couples (11), is equal to the number of Tall, Blue-eyed couples with Red hair (8), plus the number of Tall, Blue-eyed couples with White hair (3).

This obvious statement is the basic statement that will be modified with the following observations.

First Observation The number of Tall, Blue-eyed couples with Red hair must be less than or equal to the number of Blue-eyed couples with Red hair, because the set of all Blue-eyed couples with Red hair includes all *Tall*, Blue-eyed couples with Red hair, and also includes all *Short*, Blue-eyed couples with Red hair.

Second Observation Similarly, the number of Tall, Blue-eyed couples with White hair must be less than or equal to the number of Tall couples with White hair, because the set of all Tall couples with White hair includes all Tall, *Blue-eyed* couples with White hair, and also includes all Tall, *Green-eyed* couples with White hair.

So, we have:

> **The First Inequality** The number of Tall, Blue-eyed couples (11), is less than or equal to the number of Blue-eyed couples with Red hair (8+9), plus the number of Tall couples with White hair (3+7).

And now for something equivalent, but odd Now, we do something a bit odd. Consider a Blue-eyed couple with Red hair. That is, both people in the couple have Blue eyes, and both have Red hair. So, even though it seems odd, we could refer to such a couple as:

> A couple, where one person has Blue-eyes, and the other has Red hair.

Again, since in each couple the people have identical descriptions, a couple can be equivalently referred to by saying that both people are Blue-eyed with Red hair; or by saying that one person has Blue-eyes, and the other has Red hair. Using this general idea, we can rewrite the First Inequality as:

> **The Second Inequality** The number of couples where one is Tall, and the other has Blue-eyes, is less than or equal to the number of couples where one has Blue-eyes, and the other has Red hair; plus the number of couples where one is Tall, and the other has White hair.

Zeilenger calls this "Bell's inequality for twins" (couples in my version).

Comment: This inequality is fine. But note, it is an inequality based on the *entire* population of the town.

Now we tell a strange story One after another, each couple in the town will be separated, and one person from the couple will be examined by an observer called "A"; and the other person from the couple will be examined by an observer called "B". Furthermore, when an observer examines a person, the observer will only comment on *one* of the three attributes of that person. Sometimes an observer will comment on the person's height (Tall or Short). Sometimes the observer will comment on the person's eye color (Blue or Green); and sometimes the observer will comment on the person's hair color (Red or White).

Zeilenger next translates the Second Inequality (Bell's inequality for twins) into the language of pairs of particles, and measurement experiments. I will not do his full translation, but will adopt part of the translation, so that when an observer comments on the height of a person, they output "+" for Tall and "-" for Short. Similarly, when an observer comments on eye color, they output "+" for Blue and "-" for Green; and when they comment on hair color, they output "+" for Red hair, and "-" for White hair.

With that convention, Zeilenger's translation of the Second Inequality is:

> **The Proposed Third Inequality** The number of + + results where observer A comments on height, and observer B comments on eye color, is less than or equal to the number of + + results where observer A comments on eye color, and observer B comments on hair color, plus the number of + - results where observer A comments on height and observer B comments on hair color.

But, the Proposed Third Inequality certainly is not correct, because it says nothing about how *often* observer A comments on height, and observer B comments on eye color, etc. For example, suppose, in every experiment observer A comments on height, and observer B comments on eye color. Then the left side of the inequality can be something larger than zero, but the right side will always be zero.

Put another way, the number of + + results where observer A comments on height, and observer B comments on eye color, depends not only on the distribution of attributes in the population, but also on the frequency that observer A comments on height, while observer B comments on eye color, etc.

So the Second Inequality (Bell's inequality for twins) is correct, since it is a statement about the entire population, but the Proposed Third Inequality is not correct because it involves the (undiscussed) choices of what attributes the observers look at. The only thing that Zeilenger says on the issue of choice is "... with each apparatus [observer], we can perform three different kinds of measurements [observations] on the incoming particle [arriving person]. Which of these measurements [observations] is performed is determined by the experimentalists [observers] who operate their own measurement station. The experimentalists [observers] are able to decide which of the three measurements [observations] are performed." The words in square brackets are my translations of the words in Zeilenger's exposition.

How to fix this? The critical thing that is missing from Zeilenger's exposition is that in the actual experiment, for each couple, the attribute that an observer comments on is chosen *at random*, with *equal* probability that it is height, eye color, or hair color.

When we add in this detail to the story, it follows that for any couple, the probability that observer A comments on height, while observer B comments on eye color, is equal to the probability that observer A comments on eye color, while observer B comments on hair color; which is same as the probability that observer A comments on height, while observer B comments on hair color.

This leads to a correct statement:

> **Modified Third Inequality** If, for each couple, the attribute that an observer comments on is chosen from $\{height, eyecolor, haircolor\}$ with equal probability, then the probability of a + + result where observer A comments on height, and observer B comments on eye color, is less than or equal to the probability of a + + result where observer A comments on eye color, and observer B comments on hair color; plus the probability of a + - result where observer A comments on height and observer B comments on hair color.

Further, the Modified Third Inequality implies that the Proposed Third Inequality becomes more reasonable (more likely to be a correct empirical summary of the observations) as the number of couples that are examined increases, assuming again that for each couple, the attribute an observer looks at is chosen from $\{height, eyecolor, haircolor\}$ with equal probability.

But, no matter how reasonable the Proposed Third Inequality becomes, it can never be considered to be "proved", especially if we do *not* include the assumption that the choice of attribute to be commented on, is random.

Question: How to explain all this to a lay audience, who have never studied probability? We can assume a general understanding of the meaning of "at random", or "equal probability", but it is harder to explain the Modified Third Inequality, and how it implies the reasonableness of the Proposed Third Inequality. Bummer - so I probably will not include this discussion in the book.