

## 1. Introduction

- 1.1. If you flipped two coins 50 times, how many times would you get exactly one head? What is the “probability” of having two heads?
- 1.2. Computer scientists rely on probability to develop algorithms for dealing with problems such as network collisions, resource allocation, and CPU branch prediction.
- 1.3. Definition: A “sample space” is the set of all possible outcomes of an experiment or operation, e.g. {HH, HT, TH, TT}.
- 1.4. Definition: An “event” is a subset of the sample space, e.g. {HT, TH}.
- 1.5. Definition: If  $S$  is a finite sample space in which all outcomes are equally likely and  $E$  is an event in  $S$ , then the “probability of  $E$ ”, denoted  $P(E)$  is  $\frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}$ , with  $0 \leq P(E) \leq 1$ .

1.5.1. Example. What is the probability of flipping exactly one head?  $P(E) = \frac{n\{HT, TH\}}{n\{HH, HT, TH, TT\}} = \frac{2}{4} = \frac{1}{2} = 50\%$ .

1.5.2.  $P(E) = 1 - P(E^C)$  is useful when  $n(E)$  is more difficult to calculate than  $n(E^C)$ .

1.5.2.1. Example: What is the probability of having at least one of the coins being a head?

$$P(E) = 1 - \frac{n\{TT\}}{n\{HH, HT, TH, TT\}} = 1 - \frac{1}{4} = \frac{3}{4} = \frac{n\{HT, TH, HH\}}{n\{HH, HT, TH, TT\}}$$

- 1.6. Thus, to compute a probability we must properly count the number of possible outcomes for a given set. This chapter provides methods to count outcomes based on the properties of the events and sample spaces.
2. Basic Counting Principles: Given events  $E$ ,  $F$ , and  $E \cap F = \emptyset$ , i.e. disjoint, or independent events. Let  $E$  be the event of having at exactly one head when flipping two coins, and let  $F$  be the event of drawing a queen from a deck of 52 cards.

- 2.1. Product (or Sequential) Rule:  $E$  and (then)  $F$  can occur in  $n(E) * n(F)$  regardless in order, e.g. we can flip exactly one head and then draw a queen in  $2 * 4 = 8$  ways.  $P(E \text{ and } F) = \frac{2}{4} * \frac{4}{52} = \frac{1}{26} = 3.8\%$ .
- 2.2. Sum (or Disjunctive) Rule:  $E$  or  $F$  can occur in  $n(E) + n(F)$  regardless of order, e.g. we can flip exactly one head or draw a queen in  $2 + 4 = 6$  ways.  $P(E \text{ or } F) = 1 - P(E^C \text{ and } F^C) = 1 - \left(\frac{2}{4} * \frac{48}{52}\right) = 1 - \frac{24}{52} = \frac{28}{52} = \frac{7}{13} = 54\%$

## 3. Mathematical Functions

3.1. Factorial,  $n!$ , with  $0! = 1$ .

3.2. Binomial Coefficients  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$  is the number of ways  $n$  items can be chosen  $r$  at a time, i.e. “ $n$  choose  $r$ ”.

3.2.1. Binomial Theorem:  $(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i$  creates Pascal’s Triangle of coefficients:

$n$									
0			1				$= 1 * 1 * 1$		
1			1	1			$= a + b$		
2			1	2	1		$= a^2 + 2ab + b^2$		
3			1	3	3	1	$= a^3 + 3a^2b + 3ab^2 + b^3$		
4			1	4	6	4	1	$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	
5			1	5	10	10	5	1	$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

4. Permutations = any arrangement of a set of  $n$  objects in a given order is called a “permutation” of the set. Any arrangement of  $r \leq n$  of these objects in a given order is called a “permutation of the  $n$  objects taken  $r$  at a time.”

4.1.  $P(n, r) = \frac{n!}{(n-r)!} = n * (n-1) * (n-2) * \dots * (n-r+1)$ .

4.1.1. Example with four items taken two at a time  $= \frac{4!}{(4-2)!} = \frac{24}{2} = 12 = 4 * 3$ . AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC.

4.1.2. How many 5-card hands are there when order matters?  $P(52, 5) = \frac{52!}{(52-5)!} = 52 * 51 * 50 * 49 * 48 = 311,875,200$ .

4.2. With  $r = n$ , we see that there are  $n!$  permutations of  $n$  items when taken  $n$  at a time.

4.3. If there are duplicated items, then the permutation is divided by the permutation of the duplicated items. So with

$$\text{counts } n_1, n_2, n_3, \dots, n_k, \text{ then } P(n; n_1, n_2, n_3, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!}$$

4.3.1. Example: In the card game War, suits are ignored, how many permutations of such a deck are there?  $\frac{52!}{(4!)^{13}}$ .

4.4. Sampling with replacement = the element chosen is replaced in the set before the next element is chosen, e.g. the card is put back in the deck. This will just be the product rule of  $n$  elements taken  $r$  times, so  $n^r$ .

4.5. Sampling without replacement is simply normal permutations, so  $P(n, r) = \frac{n!}{(n-r)!}$ .

4.6. Example: Four cards are chosen one after the other from a 52-card deck. There are  $52^4$  ways to do this if we have replacement. If we do not have replacement after each step, then we have  $\frac{52!}{(52-4)!} = 52 * 51 * 50 * 49$  ways.

5. Combinations = taking  $n$  items  $r$  at a time, but ignoring the different orderings,  $C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$

5.1. Note that since  $C(n, r) = \frac{P(n, r)}{r!}$  there must be  $r!$  different ordering of  $r$  items, which was stated in 4.2.

5.2. Example: How many different 5-card hands are there?  $C(52, 5) = \frac{52!}{5!(52-5)!} = \frac{52*51*50*49*48}{5*4*3*2} = 13 * 17 * 5 * 49 * 48 = 2,598,960$ . Note that  $2,598,960 * 5! = 311,875,200 = P(52, 5)$

6. The Pigeonhole Principle

6.1. If  $m$  pigeons are put into  $m$  pigeonholes, there is an empty hole if and only if there's a hole with more than one pigeon. Or if  $n > m$  pigeons are put into  $m$  pigeonholes, there's at least one hole with more than one pigeon.

6.2. Example Consider a chess board with two of the diagonally opposite corners removed. Is it possible to cover the board with pieces of domino whose size is exactly two board squares?

6.2.1. No, it's not possible. Two diagonally opposite squares on a chess board are of the same color. Therefore, when these are removed, the number of squares of one color exceeds by 2 the number of squares of another color. However, every piece of domino covers exactly two squares and these are of different colors. Every placement of domino pieces establishes a one-to-one correspondence between the set of white squares and the set of black squares. If the two sets have different number of elements, then, by the Pigeonhole Principle, no 1-1 correspondence between the two sets is possible.

6.3. Example: Prove that however one selects 55 integers  $1 \leq x_1 < x_2 < x_3 < \dots < x_{55} \leq 100$ , there will be at least two that differ by 9, at least two that differ by 10, at least a pair that differ by 12, and at least a pair that differ by 13. Surprisingly, there need not be a pair of numbers that differ by 11.

6.3.1. For differing by 9, we would arrange 9 consecutive numbers 1 to 9, but then need a gap of 9, so the next would be 19-27, continuing with 37-45, 55-63, 73-81, 91-99. This uses 54 numbers, with no place for the 55<sup>th</sup> except in some hole less than 9 from some other entry. Only by choosing 100 will it differ from only one other number. All other choices will differ by 9 from two numbers. For 10, we have 1-10, 21-30, 41-50, 61-70, 81-90 which leaves 5 numbers and all remaining holes ten away from an existing number. For 11, we have 1-11, 23-33, 45-55, 67-77, and 89-100, which uses all 55 numbers. Twelve can have 1-12, 25-36, 49-60, 73-84, 97-100, which accounts for 52 of the 55. Thirteen can have 1-13, 27-39, 53-65, 79-91, which accounts for 52 of the numbers. Note that once we can have  $\lceil 55/4 \rceil = 14$  numbers in a group there is no problem, e.g. 1-14, 29-42, 57-71, 86-99 allows for 56 numbers.

6.4. Example: Prove that if  $n$  is odd then for any permutation  $p$  of the set  $\{1, 2, \dots, n\}$  the product  $P(p) = (1 - p(1))(2 - p(2)) \dots (n - p(n))$  is necessarily even. Hint: A product of several factors is even if at least one of the factors is even.

6.4.1. Solution: Since A product of several factors is even if at least one of the factors is even, I want to show that at least one term in  $P(p)$  is even. Since  $n$  is odd, and the set  $\{1, 2, \dots, n\}$  starts with an odd number, the set will have  $\frac{n+1}{2}$  odd numbers and  $\frac{n-1}{2}$  even numbers in it. Similarly, there will be  $\frac{n+1}{2}$  terms in  $P$  that subtract from an odd number, and  $\frac{n-1}{2}$  terms that subtract from an even number. The difference of two odd numbers is always an even number. To avoid producing an even term we would have to have every odd number in the set subtracted from an even number. However, since there are only  $\frac{n-1}{2}$  terms subtracting from an even

number, and we have  $\frac{n+1}{2}$  odd numbers, at least one of the odd number will be subtracted from an odd number, and produce an even term. QED.

7. The Inclusion-Exclusion Principle for sets:

7.1.  $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

7.2. Example: There are 130 ECS 20 students, 140 ECS 40 students, and 220 ECS 30 students. There are 50 students taking both ECS 20 and 40. There are 10 students taking ECS 20 and 30. There are two students taking ECS 30 and 40. There is one student taking all three courses. What is the probability that a student enrolled in at least one of the three courses is enrolled in exactly two of the courses?

7.2.1. Solution:  $P(E) = \frac{n(E)}{n(S)}$ , where  $E$  is the event that a student is taking exactly two of the courses, and  $S$  is the set of students enrolled in the courses.  $n(E) = 50 + 10 + 2 - 3 = 59$ , where the -3 accounts for the triple counting of the student who is taking all three courses. How many different students are enrolled in the three ECS courses?

$n(A \cup B \cup C) = 130 + 140 + 220 - 50 - 10 - 2 + 1 = 429$ . So  $P(E) = \frac{59}{429} = 14\%$

8. Tree Diagrams (skipped)