0. Introduction
   0.1. Both discovery and proof are integral parts of problem solving. The "discovery" is thinking of possible solutions, and the proving ensures that the proposed solution actually solves the problem.
   0.2. A "proof" is an argument or evidence establishing the truth of a proposition.
   0.3. The propositions can deal with almost anything, e.g. properties of numbers, verifying that a computer program produces a result that matches specifications, or phenomena.
   0.4. The objects of proofs are premises, conclusions, axioms (assumed true, but unprovable), definitions, and evidence from the real world.
   0.5. The techniques to manipulate these objects include the rules of inference, laws of logical equivalence, and theorems (propositions derived earlier from axioms). Different methods combine these techniques in different ways to create valid arguments.
   0.6. When writing proofs you should provided the reason for each step that is not obvious to the reader, i.e. algebraic manipulations need not be explained.
   0.7. Common definitions used in this handout:
      0.7.1. An integer n is "even" if and only if n equals twice some integer, i.e. $n = 2k$, where k is an integer.
      0.7.2. If integer n is not even, then it is "odd," and $n = 2k + 1$, where k is an integer.
1. Direct proofs
   1.1. Direct proofs prove $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ by starting with assumption that the premises $P_1, P_2, \ldots P_n$ are all true, and then applying the techniques of inference to show that Q must also be true.
   1.2. Example: Assume that if Jill has a GPA of at least 3.7, then she is on the Dean's List. Prove that Jill cannot have 3.8 GPA, and not be on the Dean's List."
   Let g = Jill has a at least a 3.7 GPA, and d = Jill is on the Dean's List, then we are trying to prove $\neg(g \wedge \neg d)$

Assertion	Reason
1. $g \rightarrow d$	premise
2. $\neg g \vee d$	(1) by logical equivalence of implication
3. $\neg g \vee \neg(\neg d)$	(2) by law of double negative
4. $\neg(g \wedge \neg d)$	(3) by De Morgans law

2. Indirect proofs rely on the negation of the consequence, $\neg q$, instead of the assuming the truth of the premise, p.
   2.1. Indirect proof by contraposition
      2.1.1. Since the implication $p \rightarrow q$ is equivalent to its contrapositive $\neg q \rightarrow \neg p$, the implication $p \rightarrow q$ can be proved by showing that its contrapositive $\neg q \rightarrow \neg p$ is true. These proofs are usually proved directly by starting at $\neg q$.
      2.1.2. Example: Prove that if x is an integer and x^2 is even, then x is even.

	Assertion	Reason
1.	x is not even	Negation of consequence
2.	$x = 2k + 1$, for some integer k	(1) by definition of odd
3.	$x^2 = (2k + 1)^2$, for some integer k	(2)
4.	$x^2 = 4k^2 + 4k + 1$, for some integer k	(3)
5.	$x^2 = 2(2k^2 + 2k) + 1$, for some integer k	(4)
6.	$2k^2 + 2k$ is an integer, for all integers k	The products and sums of integers are integers
7.	x^2 is not even	(5) and (6) by the definition of odd
8.	x is not even $\rightarrow x^2$ is not even	(1) and (7) by transitivity
9.	x^2 is even $\rightarrow x$ is even	(8) and contraposition

   2.2. Indirect proof by contradiction.
      2.2.1. Assume the proposition is false by assuming the negation of the conclusion, q, and that the premise, p, is true, and then using $p \wedge \neg q$ derive a contradiction.
      2.2.2. Proof by contradiction can be justified by noting that $(p \rightarrow q) \equiv (p \wedge \neg q \rightarrow r \wedge \neg r)$

p	q	$p \rightarrow q$	$(p \wedge \neg q) \rightarrow (r \wedge \neg r)$	$p \wedge \neg q$	$r \wedge \neg r$
T	T	T	T	F	F
T	F	F	F	T	F
F	T	T	T	F	F
F	F	T	T	F	F

2.2.3. Example: Prove that there is no integer that is both even and odd.

Assertion	Reason
1. There is at least one integer n that is both even and odd.	Negation of proposition
2. n is even	(1) by simplification
3. $n = 2k$ for some integer k	(2) by the definition of even.
4. n is odd	(1) by simplification
5. $n = 2m + 1$ for some integer m	(4) by the definition of odd.
6. $2k = 2m + 1$ for some integers k, m	(3) and (5)
7. $2k - 2m = 1$ for some integers k, m	(6)
8. $k - m = 1/2$ for some integers k, m	(7)
9. The assertion of (1) must be false	(8) is a contradiction because the difference between two integers must be an integer.
10. The original proposition must be true.	(9) and proof by contradiction.

3. Equivalence proofs (or "if-and-only-if proof", necessary-and-sufficient proof") have two methods
   3.1. Truth table.
   3.2. Use direct or indirect methods and the tautology $(p \leftrightarrow q) \equiv [(p \rightarrow q) \wedge (q \rightarrow p)]$. That is, the proposition " p if and only if q" can be proved if both the implication "if p, then q" and "if q, then p" are proved.
4. Specialized Quantified Statement Proof Techniques
   4.1. Use instantiation to access inference rules, laws of equivalence, axioms, etc., and then use generalization to return to the original statement.
   4.2. Negated universal ($\neg\forall$) and existential ($\exists$) statements can use Constructive Proofs of Existence to demonstrate the existence of an object with certain properties by creating or providing a method for creating such an object.
      4.2.1. Example: If x and y are integers, prove that $\exists$ an integer n such that $6x + 18y = 3n$.
         Let $n = 2x + 6y$, then n is an integer because it is the sum of the products of integers, and $3n = 6x + 18y$.
   4.3. Universal ($\forall$) and negated existential ($\neg\exists$) statements can be disproved by providing a single counterexample.
      4.3.1. Example: Disprove $\forall$ integers x, y, if $x^2 = y^2$ then $x = y$. Let $x = -2$, and $y = 2$, then $x^2 = 4 = 4 = y^2$, but $x \neq y$.
   4.4. "Mathematical Induction" is widely used to prove propositions of the form $\forall x\, P(x)$.
      4.4.1. A proof by mathematical induction consists of two steps:
         4.4.1.1. Basis step. The proposition $P(1)$ is shown to be true. Note that the basis step starts at the lower bound of the proposition, which may not necessarily be 1, e.g. to prove $2^n > n^2$ for $n > 4$ the basis is $P(5)$.
         4.4.1.2. Inductive step. The implication $P(k) \rightarrow P(k + 1)$ is shown to be true for every positive integer $k \geq$ base value, where the assumption that $P(k)$ is true is called the "inductive hypothesis", and $P(k + 1)$ is called the "inductive conclusion."
      4.4.2. When we complete both steps of a proof by mathematical induction, we have shown that $\forall x\, P(x)$ is true.
      4.4.3. Expressed as propositional logic, this proof technique can be stated as $[P(1) \wedge \forall k(P(k) \rightarrow P(k + 1))] \rightarrow \forall n P(n)$
      4.4.4. Typically you use the direct method and an inductive step to prove inductive conclusion by starting with one side of the conclusion and transforming it until it contains the other side. Beware of moving from both sides towards a middle statement. That often introduces an assumption of what you are trying to prove.
      4.4.5. Example: Prove that the sum of the first n odd positive integers is n^2, i.e., $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
         Basis step: $P(1)$ is $1 = 1^2 = 1$, so true.
         Inductive Hypothesis: $1 + 3 + 5 \ldots + (2k - 1) = k^2$
         Inductive Conclusion: $1 + 3 + 5 \ldots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$
         $1 + 3 + 5 \ldots + (2k - 1) + (2(k + 1) - 1) = [1 + 3 + 5 \ldots + (2k - 1)] + (2(k + 1) - 1)$ by Associative law
         $= k^2 + (2(k + 1) - 1)$ by the Inductive Hypothesis
         $= k^2 + (2k + 2 - 1)$
         $= k^2 + 2k + 1$
         $= (k + 1)^2$ which proves the inductive conclusion. Therefore the proposition is true by induction.
5. Proof by Cases divides the domain into smaller domains (cases), and then proves that the proposition is true for each case. Different methods may be used to prove the different cases.
   5.1. Prove that if n is an integer, then $n^2 \geq n$.
      Case (1): When $n = 0$, $n^2 = 0$, and $0 \geq 0$.
      Case (2): When $n \leq -1$, $n^2 > 0 > n$.
      Case (3): When $n \geq 1$, multiply both sides by n we get $n * n = n^2 \geq n * 1 = n$.
      Since the three cases address the whole domain of integers, and are shown true, the proposition is true by Proof by Cases.
   5.2. Proof by Exhaustion is a special case of Proof by Cases where every possibility is examined.
6. Strategies for selecting proof methods.
   6.1. First replace terms with their definitions, and analyze what the hypotheses and conclusions mean.
   6.2. If the statement is quantified or an equivalence, then try to apply their respective special methods.
   6.3. In general, first explore the direct method, then contraposition, and finally proof by contradiction.
   6.4. If you find a method that works for some definable class of the objects but not others, then consider proof by cases.